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Second-order wave diffraction by a vertical cylinder

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The main difficulty in second-order diffraction analysis stems from the contribution of the second-order potential, which obeys an inhomogeneous free-surface boundary condition. In some applications it is sufficient to know the second-order hydrodynamic force, which can be calculated without explicitly evaluating this second-order potential. This technique cannot however be used for calculating other quantities such as the hydrodynamic pressure at any point, the sectional force and bending moment in the cylinder or the free-surface elevation due to the second-order effects. This paper provides a detailed analysis of the second-order diffraction problem of a uniform vertical circular cylinder in regular waves. This furnishes results not only on the cylinder surface, but also on the free surface, and in principle in the fluid domain surrounding the body. The analysis may help to throw some light on the physical interpretation of the second-order theory and its mathematical description. Moreover, this information is intended to complement the development of general numerical methods for arbitrary bodies.

1. Introduction

Derivation of an analytical solution for the second-order potential due to diffraction by a vertical circular cylinder is an important step in checking numerical methods developed for arbitrary bodies. To the authors' knowledge, however, no complete solution has so far been obtained. Previous diffraction studies either assumed an incorrect radiation boundary condition, or failed to satisfy the inhomogeneous free-surface boundary condition. An expression for the second-order diffraction force due to regular waves in water of infinite depth has however been derived, by Lighthill (1979). He obtained an explicit expression for the second-order force without deriving explicitly the second-order potential; rather, use was made of an assisting potential satisfying the homogeneous form of the second-order free-surface boundary condition. The corresponding analysis in finite water depth was developed by Molin (1979). Unfortunately, these formulations involved a free-surface integral which oscillated rapidly and converged slowly. Eatock Taylor & Hung (1987) subsequently overcame this difficulty by developing a closed-form expression for the evaluation of this integral in the far field.

An alternative approach to the second-order problem is to employ an integral equation. Through the application of Green's theorem and use of a double-frequency Green function, Garrison (1984) formulated the second-order problem in regular waves by expressing the velocity potential as a distribution of wave sources and dipoles over the body surface and the free surface. This formulation is similar to the method of using an integral equation to solve the first-order problem, with the exception of an additional free-surface integral. More recently the integral equation formulation was re-examined by Kim & Yue (1989) for axisymmetric bodies and by Chau & Eatock Taylor (1988) for arbitrary bodies. Special attention was given to the treatment of the troublesome free-surface integral, and different methods of accelerating its convergence were demonstrated.

Because the calculation of the complete second-order solution is rather troublesome, various approximations have also been investigated. Thus Newman (1990) derived a simple approximation to estimate the second-order potential at large depths of submergence. This is particularly significant in determining the second-order vertical force on deep-draft bodies, but results have only been obtained for some simple geometries.

Lighthill's idea was subsequently extended by Eatock Taylor, Hung & Chau (1989). By defining the assisting potential in a suitable manner, they showed how the exact distribution of the second-order potential on the submerged surface of a body can be calculated without solving the complete second-order problem. The idea has been applied to a vertical circular cylinder, for which a closed-form expression has been derived. That formulation, however, can only provide the distribution of the potential on the body surface. In the present paper we show how one can derive a closed-form expression which furnishes results not only on the body surface but in the fluid domain surrounding the body. A method of eigenfunction expansions is used. The following sections describe the theoretical background and the essential numerical steps by which this formulation is implemented. Representative second-order results are also presented.

2. Formulation in cylindrical polar coordinates

We employ a cylindrical polar system of coordinates (r, θ, z), with the origin on the undisturbed surface and z pointing upwards. Based on the standard Stokes expansion, we use first- and second-order complex velocity potentials $\phi^{(1)}$ and $\phi^{(2)}$, with corresponding time harmonic variations $\exp(-\mathrm{i}\sigma t)$ and $\exp(-2\mathrm{i}\sigma t)$. Subscripts i and s will be used to denote incident and diffracted waves respectively. The problem for a vertical surface-piercing cylinder (of radius a) in the fluid domain Ω (of constant depth d) may then be formulated as follows:

first-order potential

$$\frac{\partial^2 \phi_{\mathrm{s}}^{(1)}}{\partial r^2}+\frac{1}{r}\frac{\partial \phi_{\mathrm{s}}^{(1)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi_{\mathrm{s}}^{(1)}}{\partial \theta^2}+\frac{\partial^2 \phi_{\mathrm{s}}^{(1)}}{\partial z^2}=0 \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\frac{\partial \phi_{\mathrm{s}}^{(1)}}{\partial r}=-\frac{\partial \phi_{\mathrm{i}}^{(1)}}{\partial r} \quad \text{at} \quad r=a, \tag{2.2}$$

$$\frac{\partial \phi_{\mathrm{s}}^{(1)}}{\partial z}=0 \quad \text{on} \quad z=-d, \tag{2.3}$$

$$\frac{\partial \phi_{\mathrm{s}}^{(1)}}{\partial z}-\nu \phi_{\mathrm{s}}^{(1)}=0 \quad \text{on} \quad z=0, \tag{2.4}$$

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\partial \phi_{\mathrm{s}}^{(1)}}{\partial r}-\mathrm{i}k\phi_{\mathrm{s}}^{(1)}\right)=0; \tag{2.5}$$

second-order potential

$$\frac{\partial^2\phi_s^{(2)}}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_s^{(2)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi_s^{(2)}}{\partial\theta^2}+\frac{\partial^2\phi_s^{(2)}}{\partial z^2}=0 \quad \text{in} \quad \Omega, \tag{2.6}$$

$$\frac{\partial\phi_s^{(2)}}{\partial r}=-\frac{\partial\phi_i^{(2)}}{\partial r} \quad \text{at} \quad r=a, \tag{2.7}$$

$$\frac{\partial\phi_s^{(2)}}{\partial z}=0 \quad \text{on} \quad z=-d, \tag{2.8}$$

$$\frac{\partial\phi_s^{(2)}}{\partial z}-4\nu\,\phi_s^{(2)}=F(r,\theta) \quad \text{on} \quad z=0. \tag{2.9}$$

Here
$$\nu=\sigma^2/g, \tag{2.10}$$

and k is the wavenumber for the first-order problem. The inhomogeneous term F may be regarded as an effective pressure distribution on the free surface which is given by

$$\begin{aligned} F(r,\theta) = & -\frac{i\sigma}{2g}\phi_i^{(1)}\left[-\frac{\sigma^2}{g}\frac{\partial\phi_s^{(1)}}{\partial z}+\frac{\partial^2\phi_s^{(1)}}{\partial z^2}\right]-\frac{i\sigma}{2g}\phi_s^{(1)}\left[-\frac{\sigma^2}{g}\frac{\partial\phi_i^{(1)}}{\partial z}+\frac{\partial^2\phi_i^{(1)}}{\partial z^2}\right] \\ & -\frac{i\sigma}{2g}\phi_s^{(1)}\left[-\frac{\sigma^2}{g}\frac{\partial\phi_s^{(1)}}{\partial z}+\frac{\partial^2\phi_s^{(1)}}{\partial z^2}\right]+\frac{i\sigma}{g}\left[\left(\frac{\partial\phi_s^{(1)}}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi_s^{(1)}}{\partial\theta}\right)^2+\left(\frac{\partial\phi_s^{(1)}}{\partial z}\right)^2\right] \\ & +\frac{2i\sigma}{g}\left[\left(\frac{\partial\phi_i^{(1)}}{\partial r}\right)\left(\frac{\partial\phi_s^{(1)}}{\partial r}\right)+\left(\frac{1}{r}\frac{\partial\phi_i^{(1)}}{\partial\theta}\right)\left(\frac{1}{r}\frac{\partial\phi_s^{(1)}}{\partial\theta}\right)+\left(\frac{\partial\phi_i^{(1)}}{\partial z}\right)\left(\frac{\partial\phi_s^{(1)}}{\partial z}\right)\right]. \end{aligned} \tag{2.11}$$

In polar coordinates, the expressions for $\phi_i^{(1)}$, $\phi_i^{(2)}$, and $\phi_s^{(1)}$ are given by Mei (1983) as follows:

$$\phi_i^{(1)}=-\frac{igA}{\sigma}\frac{\cosh k(z+d)}{\cosh(kd)}\sum_{m=0}^{\infty}\epsilon_m\, i^m J_m(kr)\cos(m\theta), \tag{2.12}$$

$$\phi_i^{(2)}=-\frac{3iA^2\sigma}{8}\frac{\cosh 2k(z+d)}{\sinh^4(kd)}\sum_{m=0}^{\infty}\epsilon_m\, i^m J_m(2kr)\cos(m\theta), \tag{2.13}$$

$$\phi_s^{(1)}=\frac{igA}{\sigma}\frac{\cosh k(z+d)}{\cosh(kd)}\sum_{m=0}^{\infty}\epsilon_m\, i_m\frac{J'_m(ka)}{H'_m(ka)}H_m(kr)\cos(m\theta), \tag{2.14}$$

where A is the amplitude of the first-order wave; and $\epsilon_m=1$ for $m=0$ and $\epsilon_m=2$ otherwise. J_m and H_m are respectively the Bessel function and Hankel function of the first kind of order m. The problem then is to determine $\phi_s^{(2)}$.

3. Second-order solution based on Green's theorem

It can readily be shown that $\phi_s^{(2)}$ may be expressed in terms of various integral equations, derived from the different forms of Green's theorem. Based on Green's second identity, the integral equation can be interpreted as a representation of the velocity potential by a mixed distribution of sources and dipoles. If we use the double-frequency linear wave source Green function (defined as G) we can express $\phi_s^{(2)}$ as a sum of integrals over the body surface, the free surface and an exterior boundary in the form of a vertical cylinder at large distance from the body. By making use of the asymptotic expression for $\phi_s^{(2)}$ given by Molin (1979), and the method of stationary phase, it can be shown that the integral over the exterior surface vanishes at infinity. Thus

$$\phi_s^{(2)} = -\int_{-d}^{0} \mathrm{d}z \int_0^{2\pi} a\mathrm{d}\theta \left(\phi_s^{(2)}(\bar{r})\frac{\partial G}{\partial r}(\bar{r}_0,\bar{r}) + G(\bar{r}_0,\bar{r})\frac{\partial \phi_i^{(2)}(\bar{r})}{\partial r}\right) - \int_a^{\infty} r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, F(\bar{r})\, G(\bar{r}_0,\bar{r}), \quad (3.1)$$

in which $\bar{r}_0(r_0, \theta_0, z_0)$ is the coordinate of a representative field point and $\bar{r}(r, \theta, z)$ is the coordinate of the integration points on the boundary.

There can be a computational advantage in solving an equation such as (3.1) if the Green function can be modified to satisfy the additional boundary condition

$$\frac{\partial G}{\partial r} = 0 \quad (3.2)$$

on the body surface. The unknown term in the integrand then vanishes. Such a choice of Green function, denoted by G_b, would provide an explicit solution of the potential in terms of the incident wave and the free-surface integral, i.e.

$$\phi_s^{(2)}(\bar{r}) = -\int_{-d}^{0} \mathrm{d}z \int_0^{2\pi} a\mathrm{d}\theta\, G_b(\bar{r}_0,\bar{r})\frac{\partial \phi_i^{(2)}}{\partial r}(\bar{r}) - \int_a^{\infty} r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, G_b(\bar{r}_0,\bar{r})\, F(\bar{r}). \quad (3.3)$$

The vertical circular cylinder is one of the few geometries for which G_b can be determined explicitly, by using separation of variables in the cylindrical polar coordinate system.

4. Construction of the Green function

The Green function G_b appropriate to the present problem is the solution of the boundary-value problem defined by the following equations:

$$\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right) G_b = \frac{1}{r}\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(z-z_0) \quad \text{in} \quad \Omega, \quad (4.1)$$

$$\frac{\partial G_b}{\partial r} = 0 \quad \text{at} \quad r = a, \quad (4.2)$$

$$\frac{\partial G_b}{\partial z} = 0 \quad \text{on} \quad z = -d, \quad (4.3)$$

$$\frac{\partial G_b}{\partial z} - 4\nu G_b = 0 \quad \text{on} \quad z = 0, \quad (4.4)$$

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\partial G_b}{\partial r} - \mathrm{i}\kappa G_b\right) = 0, \quad (4.5)$$

where κ is the real root of

$$\kappa \tanh (\kappa d) = 4\nu. \tag{4.6}$$

As suggested by Chen & Hudspeth (1982), one of the approaches to constructing G_b is to determine its expansion in a series of associated eigenfunctions. The Dirac delta functions in the θ- and z-coordinates are first expanded into complete orthonormal sets of eigenfunctions in their respective domains, i.e.

$$\delta(\theta-\theta_0) = \frac{1}{2\pi}\sum_{m=0}^{\infty} \epsilon_m \cos (m(\theta-\theta_0)), \tag{4.7}$$

$$\delta(z-z_0) = \sum_{n=0}^{\infty} Z_n(\kappa_n z_0) Z_n(\kappa_n z), \tag{4.8}$$

in which the functions $Z_n(\kappa_n z)$ form an orthonormal set in the interval $-d \leqslant z \leqslant 0$ and are defined by

$$\begin{aligned} Z_n(\kappa_n z) &= \frac{1}{N_0^{\frac{1}{2}}} \cosh (\kappa(z+d)) \quad \text{for} \quad n=0 \\ &= \frac{1}{N_n^{\frac{1}{2}}} \cos (\kappa_n(z+d)) \quad \text{for} \quad n>0. \end{aligned} \tag{4.9}$$

$\kappa_n (n>0)$ are the positive real roots of

$$\kappa_n \tan (\kappa_n d) = -4\nu \tag{4.10}$$

and the normalizing constants N_n are given by

$$\begin{aligned} N_n &= \int_{-d}^{0} \cosh^2 (\kappa(z+d))\,\mathrm{d}z \\ &= \frac{\sinh (2\kappa d)+2\kappa d}{4\kappa} = \frac{d(\kappa^2-16\nu^2)+4\nu}{2(\kappa^2-16\nu^2)} \quad \text{for} \quad n=0; \end{aligned} \tag{4.11}$$

$$\begin{aligned} N_n &= \int_{-d}^{0} \cos^2 (\kappa_n(z+d))\,\mathrm{d}z \\ &= \frac{\sin (2\kappa_n d)+2\kappa_n d}{4\kappa_n} = \frac{-d(\kappa_n^2+16\nu^2)+4\nu}{-2(\kappa_n^2+16\nu^2)} \quad \text{for} \quad n>0. \end{aligned} \tag{4.12}$$

Substituting the above expressions into (4.1) gives

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right) G_b = \frac{1}{2\pi}\delta(r-r_0)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \epsilon_m Z_n(\kappa_n z_0) Z_n(\kappa_n z) \cos (m(\theta-\theta_0)). \tag{4.13}$$

A similar expansion is assumed for G_b such that

$$G_b = \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \epsilon_m g_{mn}(r, r_0) Z_n(\kappa_n z_0) Z_n(\kappa_n z) \cos (m(\theta-\theta_0)), \tag{4.14}$$

with the r-dependent coefficients $g_{mn}(r, r_0)$ to be determined.

Substituting (4.14) into (4.1), (4.2) and (4.5) yields

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}\right)-\left(\kappa_n^2+\frac{m^2}{r^2}\right)g_{mn}=\frac{1}{r}\delta(r-r_0), \tag{4.15}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}\right|_{r=a}=0, \tag{4.16}$$

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}-\mathrm{i}\kappa g_{mn}\right)=0. \tag{4.17}$$

Equation (4.15) can now be replaced by the following sets of equations:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}\right)-\left(\kappa_n^2+\frac{m^2}{r^2}\right)g_{mn}=0; \tag{4.18}$$

$$g_{mn}(r_0^+,r_0)=g_{mn}(r_0^-,r_0) \tag{4.19}$$

(i.e. the function g_{mn} is continuous at $r=r_0$);

$$\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}(r_0^+,r_0)-\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}(r_0^-,r_0)=\frac{1}{r_0} \tag{4.20}$$

(i.e. the derivative of g_{mn} has a discontinuity at $r=r_0$).

These additional end conditions across $r=r_0$ have been deduced by first integrating (4.15) throughout the small interval $(r_0-\epsilon, r_0+\epsilon)$, and then considering the limiting form of the resulting differential equation as $\epsilon\to 0$. In order to satisfy the limiting conditions at the point $r=r_0$, g_{mn} must behave in such a way that $\mathrm{d}g_{mn}/\mathrm{d}r$ has a jump of magnitude $1/r_0$ as $r\to r_0$.

In order to obtain a convenient reformulation of the problem, two equivalent boundary-value problems are defined such that, for a given r_0, g_{mn} is given by g_{mn}^{I} when $r<r_0$, and by g_{mn}^{II} when $r>r_0$. These have the following properties:
boundary-value problem I ($a\leqslant r<r_0$)

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}^{\mathrm{I}}\right)-\left(\kappa_n^2+\frac{m^2}{r_2}\right)g_{mn}^{\mathrm{I}}=0, \tag{4.21}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}^{\mathrm{I}}\right|_{r=a}=0; \tag{4.22}$$

boundary-value problem II ($r_0<r<\infty$)

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}^{\mathrm{II}}\right)-\left(\kappa_n^2+\frac{m^2}{r^2}\right)g_{mn}^{\mathrm{II}}=0, \tag{4.23}$$

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}^{\mathrm{II}}-\mathrm{i}\kappa g_{mn}^{\mathrm{II}}\right)=0; \tag{4.24}$$

and with the matching conditions at $r=r_0$

$$g_{mn}^{\mathrm{I}}(r_0)=g_{mn}^{\mathrm{II}}(r_0), \tag{4.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}^{\mathrm{II}}(r_0)-\frac{\mathrm{d}}{\mathrm{d}r}g_{mn}^{\mathrm{I}}(r_0)=\frac{1}{r_0}. \tag{4.26}$$

By solving the above ordinary differential equations with the prescribed boundary conditions, one obtains

$$g^{\mathrm{I}}_{mn} = -\frac{K_m(\kappa_n r_0)}{K'_m(\kappa_n a)}[I_m(\kappa_n r)K'_m(\kappa_n a) - I'_m(\kappa_n a)K_m(\kappa_n r)], \tag{4.27}$$

$$g^{\mathrm{II}}_{mn} = -\frac{K_m(\kappa_n r)}{K'_m(\kappa_n a)}[I_m(\kappa_n r_0)K'_m(\kappa_n a) - I'_m(\kappa_n a)K_m(\kappa_n r_0)]. \tag{4.28}$$

Here the prime denotes differentiation with respect to the argument, and I_m and K_m are the modified Bessel functions of the first and second kinds respectively. When $n = 0$, $\kappa_n = \mathrm{i}\kappa$: the corresponding functions of real arguments κr etc. are J_m and $H^{(1)}_m$. Use has here been made of the Wronskian relation

$$K_p(x)I'_p(x) - I_p(x)K'_p(x) = \frac{1}{x}. \tag{4.29}$$

After substitution of (4.27) and (4.28) into (4.14), the expression for G_{b} may be written in a compact form as

$$G_{\mathrm{b}} = \frac{-1}{2\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\epsilon_m \frac{K_m(\kappa_n r_>)}{K'_m(\kappa_n a)}[I_m(\kappa_n r_<)K'_m(\kappa_n a) - I'_m(\kappa_n a)K_m(\kappa_n r_<)]$$
$$Z_n(\kappa_n z_0)\,Z_n(\kappa_n z)\cos(m(\theta-\theta_0)) \tag{4.30}$$

where $$r_> = \max(r_0, r), \quad r_< = \min(r_0, r). \tag{4.31}$$

5. Expression for the second-order diffracted potential

By substitution of (4.30) for G_{b} into the integral equation (3.3), the second-order diffracted potential may be recovered as follows:

$$\begin{aligned}\phi^{(2)}_{\mathrm{s}}(\bar{r}_0) = {} & \frac{1}{2\pi}\int_a^{\infty} r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\,F(r,\theta)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\epsilon_m Z_n(\kappa_n z_0)Z_n(0)\cos(m(\theta-\theta_0))\\ & \times\frac{K_m(\kappa_n r_>)}{K'_m(\kappa_n a)}[I_m(\kappa_n r_<)K'_m(\kappa_n a) - I'_m(\kappa_n a)K_m(\kappa_n r_<)]\\ & + \frac{1}{2\pi}\int_{-d}^{0}\mathrm{d}z\int_0^{2\pi} a\,\mathrm{d}\theta\,\frac{\partial\phi^{(2)}_{\mathrm{i}}}{\partial r}(a,\theta,z)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\epsilon_m Z_n(\kappa_n z_0)Z_n(\kappa_n z)\cos(m(\theta-\theta_0))\\ & \times\frac{K_m(\kappa_n r_0)}{K'_m(\kappa_n a)}[I_m(\kappa_n a)K_m(\kappa_n a) - I'_m(\kappa_n a)K_m(\kappa_n a)].\end{aligned} \tag{5.1}$$

After inserting the Fourier cosine series expressions for F and $\phi^{(2)}_{\mathrm{i}}$, making use of the Wronskian relation and employing the orthogonality of the eigenfunctions of θ and z, one can reduce (5.1) to

$$\begin{aligned}\phi^{(2)}_{\mathrm{s}}(\bar{r}_0) = {} & \sum_{m=0}^{\infty}\left\{\int_a^{\infty} r\,\mathrm{d}r\,\epsilon_m F_m(r)\sum_{n=0}^{\infty}Z_n(\kappa_n z_0)Z_n(0)\right.\\ & \left.\times K_m(\kappa_n r_>)\left[I_m(K_n r_<) - \frac{I'_m(\kappa_n a)}{K'_m(\kappa_n a)}K_m(\kappa_n r_<)\right]\right\}\cos(m\theta_0)\\ & - \sum_{m=0}^{\infty}\left\{\epsilon_m D'_m\Big|_{r=a}\sum_{n=0}^{\infty}Z_n(\kappa_n z_0)Z_n(0)Z_{3n}\frac{K_m(\kappa_n r_0)}{\kappa_n K'_m(\kappa_n a)}\right\}\cos(m\theta_0).\end{aligned} \tag{5.2}$$

The following definitions are used:

$$F(r,\theta) = \sum_{m=0}^{\infty} \epsilon_m F_m(r) \cos m\theta; \tag{5.3}$$

$$D_m = -\frac{\mathrm{i}3A^2\sigma}{8\sinh^4(kd)} \mathrm{i}^m J_m(2kr); \tag{5.4}$$

$$D'_m\bigg|_{r=a} = \frac{\mathrm{d}}{\mathrm{d}r} D_m\bigg|_{r=a}; \tag{5.5}$$

$$\begin{aligned} Z_{3n} &= \frac{1}{N_n^{\frac{1}{2}} \cos(\kappa_n d)} \int_{-d}^{0} Z_n(\kappa_n z) \cosh(2k(z+d))\,\mathrm{d}z \\ &= \frac{2k\sinh(2kd) - 4\nu\cosh(2kd)}{4k^2+\kappa_n^2}. \end{aligned} \tag{5.6}$$

In particular, for $r_0 = a$ (i.e. on the body surface):

$$\begin{aligned} \phi_{\mathrm{s}}^{(2)}(a) = -\sum_{m=0}^{\infty} \Bigg\{ & \int_a^{\infty} r\,\mathrm{d}r\,\epsilon_m F_m(r) \sum_{n=0}^{\infty} Z_n(\kappa_n z_0) Z_n(0) \frac{K_m(\kappa_n r)}{a\kappa_n K'_m(\kappa_n a)} \\ & + \epsilon_m D'_m\bigg|_{r=a} \sum_{n=0}^{\infty} Z_n(\kappa_n z_0) Z_n(0) Z_{3n} \frac{K_m(\kappa_n a)}{\kappa_n K'_m(\kappa_n a)} \Bigg\} \cos(m\theta_0). \end{aligned} \tag{5.7}$$

It may be observed that, apart for the notation used for $F_m(r)$, the above result is identical to that obtained in a completely different manner by Eatock Taylor *et al.* (1989).

The hydrodynamic force, P, in the direction of the waves due to the second-order diffracted potential can then be evaluated as follows:

$$\begin{aligned} P &= -2\mathrm{i}\sigma\rho \int_{-d}^{0} \mathrm{d}z \int_0^{2\pi} a\,\mathrm{d}\theta\,\phi_{\mathrm{s}}^{(2)}(a) \cos\theta_0 \\ &= -2\mathrm{i}\sigma\rho a\pi \int_{-d}^{0} \mathrm{d}z\,\phi_{\mathrm{s},1}^{(2)}, \end{aligned} \tag{5.8}$$

where $\phi_{\mathrm{s},1}^{(2)}$ is the component of $\phi_{\mathrm{s}}^{(2)}$ that is proportional to $\cos\theta_0$ in (5.7), and ρ is the fluid density. After substitution of the corresponding expression for $\phi_{\mathrm{s},1}^{(2)}$, this gives

$$\begin{aligned} P &= 4\mathrm{i}\sigma\rho\pi \sum_{m=0}^{\infty} \frac{Z_n(0)}{\kappa_n K'_1(\kappa_n a)} \left(\int_a^{\infty} r\,\mathrm{d}r\,F_1(r) K_1(\kappa_n r) \int_{-d}^{0} \mathrm{d}z\,Z_n(\kappa_n z) \right) \\ &\quad + 4\mathrm{i}a\sigma\rho\pi D'_1\bigg|_{r=a} \sum_{n=0}^{\infty} Z_n(0) Z_{3n} \frac{K_1(\kappa_n a)}{\kappa_n K'_1(\kappa_n a)} \int_{-d}^{0} \mathrm{d}z\,Z_n(\kappa_n z) \\ &= 4\mathrm{i}\sigma\rho\pi \sum_{n=0}^{\infty} \frac{\cos\kappa_n d \sin\kappa_n d}{N_n \kappa_n^2 K'_1(\kappa_n a)} \int_a^{\infty} r\,\mathrm{d}r\,F_1(r) K_1(\kappa_n r) \\ &\quad + 4\mathrm{i}a\sigma\rho\pi D'_1\bigg|_{r=a} \sum_{n=0}^{\infty} \frac{\cos(\kappa_n d)\sin(\kappa_n d) Z_{3n} K_1(\kappa_n a)}{N_n \kappa_n^2 K'_1(\kappa_n a)}. \end{aligned} \tag{5.9}$$

This is identical to the corresponding expression for wave force given by Eatock Taylor & Hung (1987).

In view of the controversy that has surrounded earlier attempts to solve the second-order diffraction problem, it is appropriate to examine the solution given in

(5.2) in the context of previous deficiencies. The crucial limitation of most solutions published hitherto is that they do not appear to satisfy the inhomogeneous free-surface boundary condition (2.9) which characterizes the second-order problem. We need to show that the expression for $\phi_s^{(2)}$ introduced in (5.2) does satisfy (2.9) in a limiting sense. Suppose we define the function

$$Y(z) = \frac{\partial \phi_s^{(2)}}{\partial z} - 4\nu \phi_s^{(2)}. \tag{5.10}$$

It seems that the expression for $\phi_s^{(2)}$ given in (5.2) contradicts the inhomogeneous free-surface boundary condition (2.9), because it would appear that $Y(0) = 0$. The contradiction can be explained, however, by the fact that the series representation of $Y(z)$ is not a continuous function of z at $z = 0$. The series solution given in (5.2) does not yield a uniformly convergent result for $Y(z)$ as z approaches zero: but the series representation of $Y(z)$ does approach $F(r, \theta)$. Thus by means of arguments such as given by Friedman (1956), expression (5.2) can indeed be shown to satisfy the inhomogeneous free-surface boundary condition. Details are provided in Appendix A.

6. Numerical implementation

The second-order diffracted potential can be evaluated explicitly from expression (5.2). It is clear, however, that owing to the complexity of the infinite radial line integrals associated with the inhomogeneous free-surface boundary condition, special numerical procedures have to be used. These are summarized here, with further details given in Appendices B and C.

The starting point for the integration is use of a Gaussian 3-point adaptive technique. In order to speed up convergence, however, numerical quadrature is only employed up to a finite range and this is complemented by an analytical integration to infinity. The infinite integral can be evaluated by substituting Hankel's asymptotic expansions in the integrand, whose typical terms are then triple products of Bessel and Hankel functions. Eventually, the integrand can be represented by summations of polynomials of various orders. Integration of each term of the polynomial satisfies a simple recurrence relationship from which its value can be easily calculated.

We first rewrite the inhomogeneous term in the second-order free-surface boundary condition (2.11), using the linearized free-surface boundary condition (2.4) and the following relationship (which in the case of a vertical circular cylinder holds for all z):

$$\frac{\partial^2 \phi_s^{(1)}}{\partial z^2} = k^2 \phi_s^{(1)}. \tag{6.1}$$

We thus obtain

$$\begin{aligned} F(r,\theta) = {} & \frac{\mathrm{i}\sigma}{2g} k^2 (3\tanh^2(kd) - 1)\,[\phi_s^{(1)}\phi_s^{(1)} + 2\phi_i^{(1)}\phi_s^{(1)}] \\ & + \frac{\mathrm{i}\sigma}{g}\left[\left(\frac{\partial \phi_s^{(1)}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \phi_s^{(1)}}{\partial \theta}\right)^2\right. \\ & \left. + 2\left(\frac{\partial \phi_i^{(1)}}{\partial r}\frac{\partial \phi_s^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial \phi_i^{(1)}}{\partial \theta}\frac{\partial \phi_s^{(1)}}{\partial \theta}\right)\right]. \end{aligned} \tag{6.2}$$

$\phi_{\rm i}^{(1)}$ and $\phi_{\rm s}^{(1)}$ are given by (2.12) and (2.14) respectively, but they can be expressed more conveniently by

$$\phi_{\rm i}^{(1)}(r,\theta,z) = \sum_{m=-\infty}^{\infty} T_m(r,z)\,{\rm e}^{{\rm i}m\theta}, \tag{6.3}$$

$$\phi_{\rm s}^{(1)}(r,\theta,z) = \sum_{m=-\infty}^{\infty} S_m(r,z)\,{\rm e}^{{\rm i}m\theta}. \tag{6.4}$$

Here

$$T_m(r,z) = \frac{-{\rm i}gA}{2\sigma}\frac{\cosh\,(k(z+d))}{\cosh\,(kd)}\,{\rm i}^m\,\{H_m(kr)+H_m^*(kr)\} \tag{6.5}$$

(where * denotes complex conjugate); and

$$S_m(r,z) = \frac{{\rm i}gA}{\sigma}\frac{\cosh\,(k(z+d))}{\cosh\,(kd)}\frac{J_m'(ka)}{H_m'(ka)}H_m(kr). \tag{6.6}$$

Examination of (6.2) reveals that its typical elements contain quadratic products of the linearized velocity potentials and their derivatives: they are of the form

$$\phi_{\rm i}^{(1)}\phi_{\rm s}^{(1)} = \sum_{p=-\infty}^{\infty} T_p\,{\rm e}^{{\rm i}p\theta}\sum_{q=-\infty}^{\infty} S_q\,{\rm e}^{{\rm i}q\theta}. \tag{6.7}$$

In order to identify harmonics in θ, the double summation in the above expression is rearranged to give

$$\sum_{p=-\infty}^{\infty} T_p{\rm e}^{{\rm i}p\theta}\sum_{q=-\infty}^{\infty} S_q{\rm e}^{{\rm i}q\theta} = \sum_{p=-\infty}^{\infty}\left\{\sum_{q=-\infty}^{\infty} T_{p-q}S_q\right\}{\rm e}^{{\rm i}p\theta} \tag{6.8}$$

and similarly for other components. With this rearrangement, (6.2) now becomes

$$F(r,\theta) = \sum_{p=-\infty}^{\infty} F_p(r)\,{\rm e}^{{\rm i}p\theta} \tag{6.9}$$

with

$$F_p(r) = F_{-p}(r); \tag{6.10}$$

and

$$\begin{aligned} F_p(r) = \sum_{q=-\infty}^{\infty}\Bigg\{&\frac{{\rm i}\sigma}{2g}k^2(3\tanh^2(kd)-1)[S_{p-q}S_q+2T_{p-q}S_q] \\ &+\frac{{\rm i}\sigma}{g}\Bigg[\frac{\partial}{\partial r}S_{p-q}\frac{\partial}{\partial r}S_q-\frac{q(p-q)}{r^2}S_{p-q}S_q \\ &+2\left(\frac{\partial}{\partial r}T_{p-q}\frac{\partial}{\partial r}S_q-\frac{q(p-q)}{r^2}T_{p-q}S_q\right)\Bigg]\Bigg\}_{z=0}. \end{aligned} \tag{6.11}$$

Returning now to the explicit expression for $\phi_{\rm s}^{(2)}$ given in (5.2), we rewrite the free-surface integral, denoted here by I_F, in terms of two line integrals such that

$$I_F = I_{F^1}+I_{F^2}. \tag{6.12}$$

In this

$$\begin{aligned} I_{F^1} = \sum_{m=0}^{\infty}\Bigg\{&\sum_{n=0}^{\infty} Z_n(\kappa_n z_0)\,Z_n(0)\,K_m(\kappa_n r_0) \\ &\times\int_a^{r_{\rm s}} r\,{\rm d}r\,F_m(r)\left[I_m(\kappa_n r)-\frac{I_m'(\kappa_n a)}{K_m'(\kappa_n a)}K_m(\kappa_n r)\right]\Bigg\}\epsilon_m\cos\,(m\theta_0), \end{aligned} \tag{6.13}$$

$$\begin{aligned} I_{F^2} = \sum_{m=0}^{\infty}\Bigg\{&\sum_{n=0}^{\infty} Z_n(\kappa_n z_0)\,Z_n(0)\left[I_m(\kappa_n r_0)-\frac{I_m'(\kappa_n a)}{K_m'(\kappa_n a)}K_m(\kappa_n r_0)\right] \\ &\times\int_{r_{\rm s}}^{\infty} r\,{\rm d}r\,F_m(r)\,K_m(\kappa_n r)\Bigg\}\epsilon_m\cos\,(m\theta_0). \end{aligned} \tag{6.14}$$

As discussed below, r_s is chosen such that quadrature is only required over a finite range (I_{F^1}), but complemented by an explicit integration to infinity (I_{F^2}).

Prior to deriving the aforementioned analytical expression for I_{F^2} one can further simplify the integrand by neglecting the contributions due to the evanescent modes ($n > 0$) and those due to the circumferential derivatives in $F_m(r)$ at large values of r. Appendix B shows how the resulting integral can be transformed, using integration by parts, so that the only remaining integral (denoted by I_{F_∞}) no longer involves radial derivatives. It may be expressed as

$$I_{F_\infty} = (3\tanh^2(kd)+k^2-\kappa^2)\int_{r_s}^{\infty} r\,\mathrm{d}r \sum_{q=-\infty}^{\infty} [S_{m-q}S_q+2T_{m-q}S_q]H_m(\kappa r). \quad (6.15)$$

Each term of the series involves two parts, which are of the form

$$I^{(1)}_{\iota mn} = \int_{r_s}^{\infty} r\,\mathrm{d}r\,H_\iota(kr)\,H_m(kr)\,H_n(\kappa r); \quad (6.16)$$

and

$$I^{(2)}_{\iota mn} = \int_{r_s}^{\infty} r\,\mathrm{d}r\,H^*_\iota(kr)\,H_m(kr)\,H_n(\kappa r) \quad (6.17)$$

respectively. The remaining task, also described in Appendix B, is to develop efficient algorithms for the explicit evaluation of the above integrals, whose integrands contain triple products of Hankel functions. This is achieved by use of Hankel's asymptotic expansions. The simplest approach would be to use the leading-order terms of the asymptotic expansions, substituting these into (6.16) and (6.17). The infinite integrals would follow in the form of Fresnel integrals and could be evaluated explicitly. In this approach, however, the chosen value of r_s would have to be very large (especially for large m), if the leading-order approximation were to yield sufficient accuracy for r greater than r_s. For smaller r_s, the leading-order terms are not sufficiently accurate by themselves. They can however be amended by residual corrections, which are based on subsequent terms of the asymptotic power series (cf. (B 6) and (B 7) in Appendix B). In this manner, more refined approximations for the integrands can be obtained.

It may be observed that the leading asymptotic in these series does not depend on the order m; the coefficients of the full asymptotic expansions do however involve m: equations (B 6) and (B 7) are not uniform with respect to m. For a given value of the argument x there exists an optimal term at which truncation of the asymptotic series yields the maximal accuracy and the function is approximated to an error within the size of this term (see Bender & Orszag 1978 for example). The accuracy cannot be improved by merely taking more terms in the partial sums, but only by increasing the value of x. Moreover, the smallest value of x at which the approximation is accurate increases with the value of m. Thus for the purpose of calculating I_{F_∞}, one has to use different values of r_s for different values of m.

In the analysis performed here, an adaptive procedure has been adopted. This enables one to overcome the apparently limited accuracy associated with the optimal truncation for given r_s, and provides any degree of accuracy as long as a sufficient number of terms of the series is used. This is because the larger r_s is, the more accurate the asymptotic series can be, because the size of the optimal truncation term is smaller. The exact location of the optimal term is not critical and the choice of r_s is made adaptively. The infinite line integral is therefore now written

$$\int_a^\infty = \int_a^{r_{s^1}} + \int_{r_{s^1}}^{r}. \quad (6.18)$$

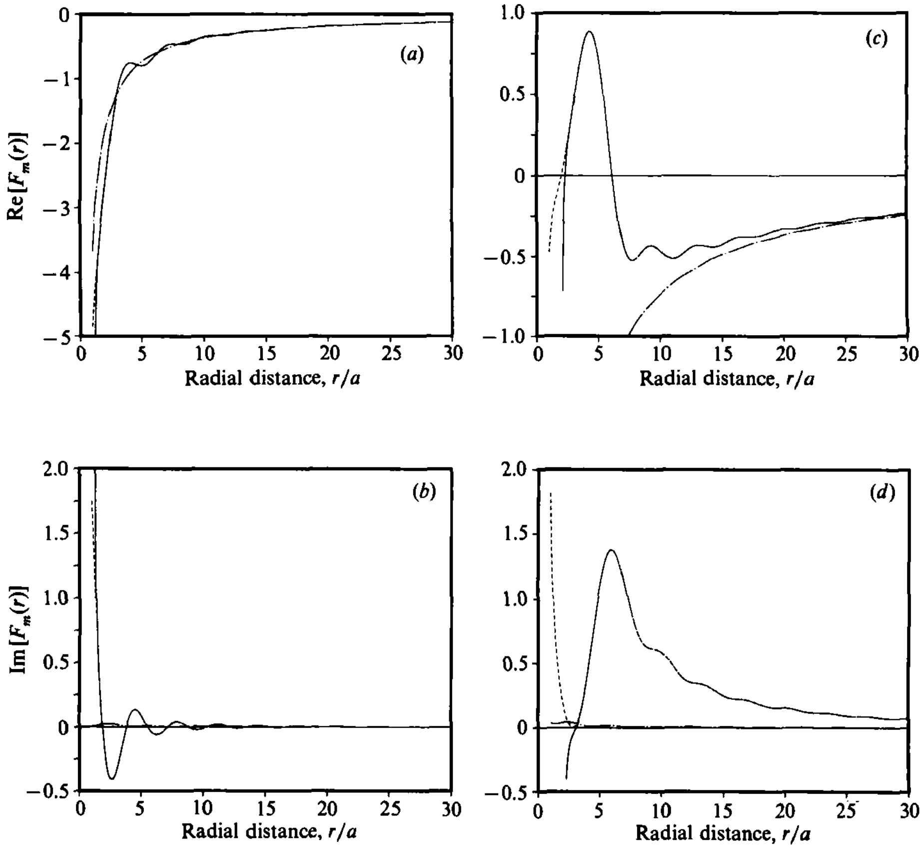


FIGURE 1. Fourier harmonics of free-surface effective pressure ($d/a = 3$, $ka = 1$): ——, analytical expression; -----, approximation based on Hankel asymptotic expansion; —·—, approximation based on first term of asymptotic expansion. (a) Re[$F_0(r)$]; (b) Im[$F_0(r)$]; (c) Re[$F_4(r)$]; (d) Im[$F_4(r)$].

The first integral on the right is computed numerically (say with the Gaussian 3-points adaptive technique) by using the exact representation for the Hankel functions; while the second integral is evaluated explicitly after substituting the Hankel asymptotic expansion. The integration limit of the finite integral is then progressively increased and the new value can be added to the integration from r_0 to r_{s^1} such that

$$\int_{r_{s^1}}^{\infty} = \int_{r_{s^1}}^{r_{s^2}} + \int_{r_{s^2}}^{\infty}. \tag{6.19}$$

The last integral of the asymptotic series is re-evaluated each time. This procedure is then repeated until convergence is obtained by comparing the values of several successive approximations. A similar idea has been used by Eatock Taylor & Hung (1987) except that in their analysis, only the leading asymptotic term was retained.

The effectiveness of using terms beyond the leading asymptotic term may be demonstrated by some calculated results for the effective pressure on the free surface. In figure 1, the exact expression for the pressure term given in (6.11) is plotted against the radial distance, and compared with the results approximated by the

(a) ι	m	n	Asymptotic expansion of integral	Gaussian quadrature
1	2	3	(0.001886, 0.003742)	(0.001886, 0.003742)
5	3	6	(−0.006758, 0.002120)	(−0.006758, 0.002120)
9	9	12	(−0.011812, −0.034072)	(−0.011812, −0.034072)
11	11	14	(0.109891, 0.000471)	(0.109891, 0.000471)
14	9	17	(−0.361132, 0.022987)	(−0.361132, 0.022987)
12	12	15	(−0.040433, −0.246565)	(0.040433, −0.246565)
13	13	17	(−0.240852, 0.712168)	(−0.240853, 0.712166)

(b) ι	m	n	Asymptotic expansion of integral	Gaussian quadrature
1	2	3	(−0.030152), −0.020504)	(−0.030152, −0.020504)
5	3	6	(−0.038240, 0.000367)	(−0.038240, 0.000367)
9	9	12	(0.029192, −0.064118)	(0.029192, −0.064118)
11	11	14	(−0.115777, 0.086181)	(−0.115777, 0.086181)
14	9	17	(0.366716, −0.028274)	(0.366716, −0.028274)
12	12	15	(0.110646, 0.261059)	(0.110646, 0.261059)
13	13	17	(0.201465, −0.757932)	(0.201465, −0.757929)

TABLE 1. Numerical verification of the asymptotic expansions of the integrals: (a) expression (6.16); (b) expression (6.17)

leading asymptotic and the asymptotic form truncated at the term M_m (defined in equation (B 8)). F_m is the mth Fourier harmonic of the free-surface effective pressure $F(r, \theta)$. Here real and imaginary components of $F_m(r)$ for $m = 0$ and $m = 4$ are plotted against r for a uniform vertical cylinder (radius $= a$) at $ka = 1$, in water of depth $3a$. It is clearly seen that the accuracy and the range of validity of the asymptotic approximation are drastically improved by including more than just the leading term in the asymptotic series. One can also observe from figure 1 that even with a small number of terms, the asymptotic series is dependable for relatively small values of r/a.

These techniques have led to an effective method for evaluating the integrals in (6.16) and (6.17). Table 1 provides some illustrative results for the case $ka = 1$, $d/a = 3$, obtained over the finite interval $[10a, 11a]$ for an increasing series of values of ι, m, n (the orders of the three Hankel functions in the integrand). Tables 1(a) and 1(b) relate to equations (6.16) and (6.17) respectively. The results are compared with those obtained by Gaussian quadrature (relative error 10^{-6}) of the exact representation of the Hankel functions. It is seen that discrepancies only occur at the sixth decimal place (at the bottom of the table). In comparison with a similar table given by Kim & Yue (1989), the present method introduces less error at the higher orders. (In their approach, Chebyshev polynomials were used to represent the Hankel functions while the resulting infinite integrals were expressed in terms of Fresnel integrals.)

To give an indication of the convergence of the procedure, calculations of I_{F_∞} are presented here for a uniform vertical cylinder for three Fourier modes ($m = 0$, 5 and 10). In this and subsequent analyses, the radius a of the cylinder is taken to be equal to the water depth d, and $ka = 1$. The results are given in table 2. Here r_{s_1} is chosen to be $2d$ and the step length Δs is given by

$$\Delta s = r_{s,j+1} - r_{s,j} = 2\pi/(2k+\kappa). \tag{6.20}$$

From the results presented, it is clear that the rate of convergence of the integral I_{F_∞}

No. of steps	I_{F_∞} $m = 0$	$m = 5$	$m = 10$
1	$(1.03647\times10^{-3}$ $-1.80242\times10^{-3})$	$(5.76233\times10^{6}$ $2.34660\times10^{6})$	$(-1.55479\times10^{15}$ $-1.81532\times10^{16})$
2	$(1.03493\times10^{-3}$ $-1.80312\times10^{-3})$	$(-0.97623$ $-9.05625)$	$(-1.79377\times10^{9}$ $1.56574\times10^{9})$
3	$(1.03507\times10^{-3}$ $-1.80310\times10^{-3})$	$(5.98646\times10^{-4}$ $-5.83335\times10^{-4})$	$(7.45227\times10^{3}$ $1.68548\times10^{3})$
4	$(1.03500\times10^{-3}$ $-1.80316\times10^{-3})$	$(5.98550\times10^{-4}$ $-5.83763\times10^{-4})$	$(-7.99964\times10^{-2}$ $-1.86592\times10^{-1})$
5	—	$(5.98513\times10^{-4}$ $-5.83722\times10^{-4})$	$(-1.23421\times10^{-5}$ $-1.58235\times10^{-5})$
6	—	$(5.98530\times10^{-4}$ $-5.83731\times10^{-4})$	$(-1.24694\times10^{-5}$ $-1.58492\times10^{-5})$
7			$(-1.24614\times10^{-5}$ $-1.58490\times10^{-5})$
8			$(-1.24593\times10^{-5}$ $-1.58505\times10^{-5})$
9			$(-1.24595\times10^{-5}$ $-1.58526\times10^{-5})$

TABLE 2. Convergence of I_{F_∞} when adaptive integration is used in conjunction with the asymptotic approximation (real part and imaginary part)

is very satisfactory. The corresponding integrals which are defined over the finite intervals may be evaluated very accurately by means of the Gaussian 3-points adaptive method. The algorithm is based on successive subdivision of each integration range into finer intervals in such a way that all integration points of the coarse interval are contained in the finer interval. Values of the integrand computed at each step of the calculation are then used in subsequent steps.

There remains just one aspect which requires careful attention. This concerns the occurrence of a weak singularity at the free surface, which would be expected to lead to slow convergence (i.e. a large number of integration points) when the field points are in the vicinity of the free surface. The nature of the singularity is established in Appendix C and it is shown how convergence of the numerical integration can be improved by subtracting out the singularity.

7. Numerical results and discussion

The availability of the closed-form expression for the second-order solution for a vertical cylinder furnishes a source of validation for the general numerical methods which are being developed for bodies of arbitrary geometry. Results obtained for this simplest three-dimensional body can also provide valuable physical insight, and enhance the understanding of nonlinear wave effects on more realistic structures in the ocean.

Although reliable results for the second-order potential and the related physical quantities are extremely scarce, it is possible to validate the present formulation by comparing results with those from a numerical method described by Kim & Yue (1989). Only terms due to the contribution of the second-order diffraction potential have been chosen for comparison, since the calculation of other terms is trivial for the case of a circular cylinder. Figures 2 (*a*) and 2 (*b*) show the pressure distributions down

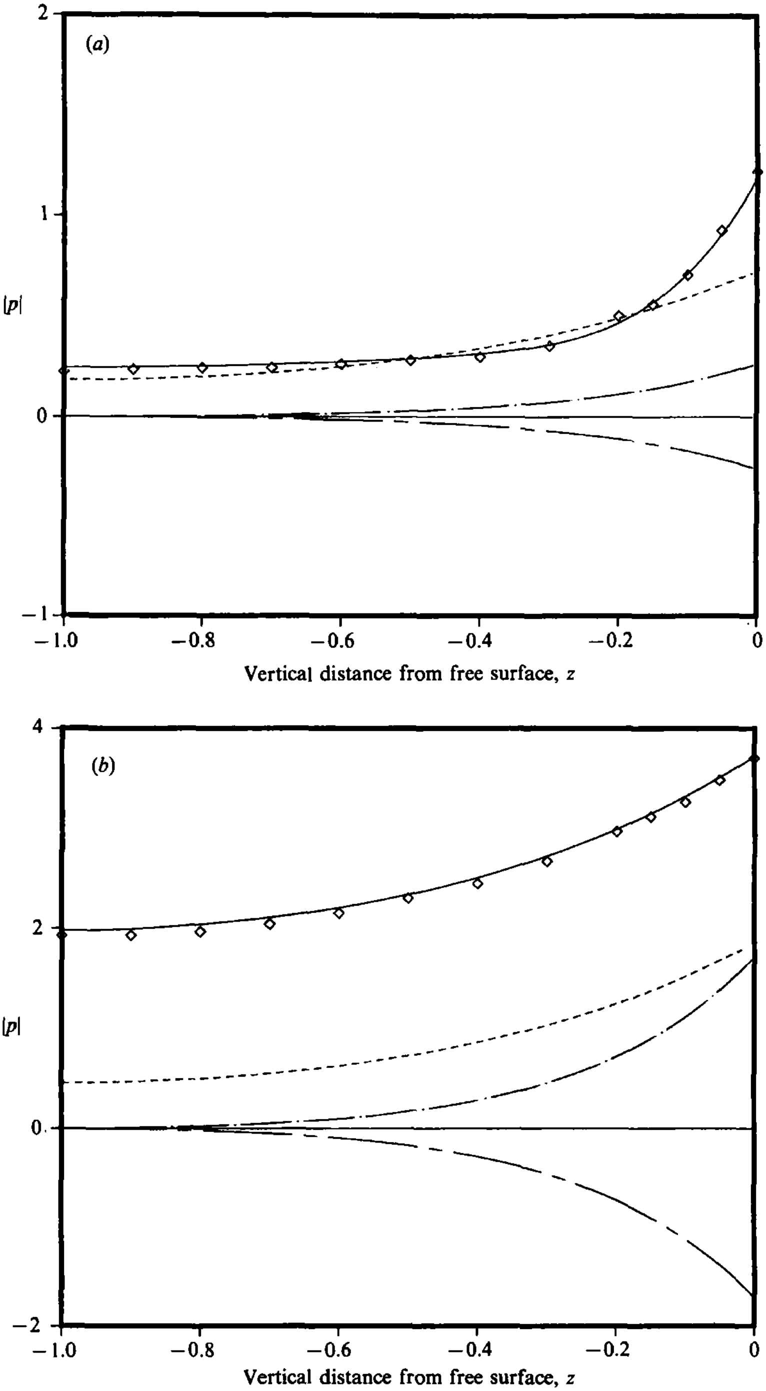


FIGURE 2. Dimensionless pressure on generator of a vertical cylinder ($d/a = 1$, $\nu a = 2$); -----, $|p^{(1)}|$; —-—, $\bar{p}^{(2)}$; —·—, $|p_1^{(2)}|$; ——, $|p_2^{(2)}|$; ◇, $|p_2^{(2)}|$ (Kim & Yue 1989). (*a*) $\theta = 0°$ (downwave); (*b*) $\theta = 180°$ (upwave).

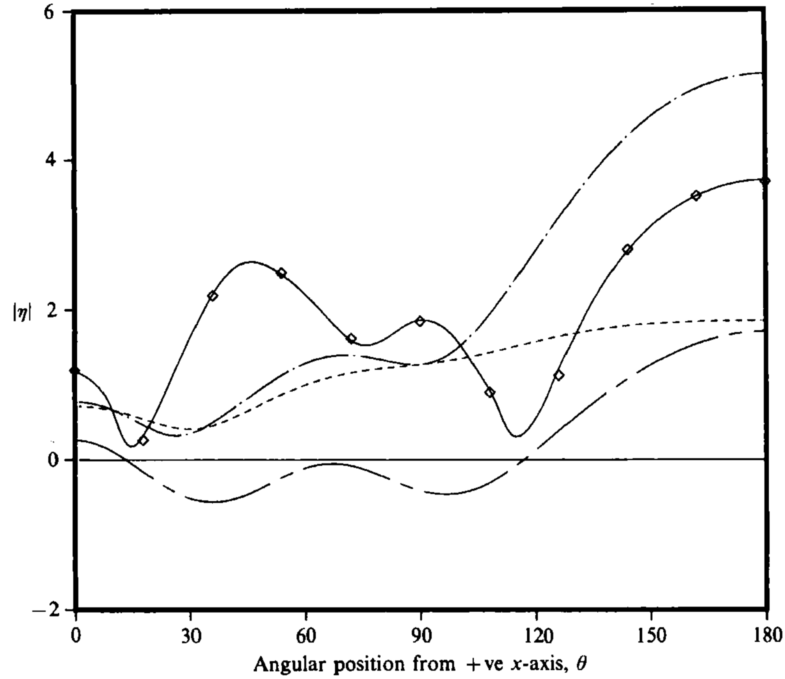


FIGURE 3. Dimensionless wave elevation around the circumference of a vertical cylinder ($d/a = 1$, $\nu a = 2$): -----, $|\eta^{(1)}|$; —·—, $\bar{\eta}^{(2)}$; —·—, $|\eta_1^{(2)}|$; ——, $|\eta_2^{(2)}|$; ◇, $|\eta_s^{(2)}|$ (Kim & Yue 1989).

the generators corresponding to $\theta = 0°$ (downwave) and 180° (upwave) respectively. For illustration, the first-order quantities and components of the second-order quantities derived from the analytical solution are also plotted in the figures. The notation, also used by Kim & Yue (1989), is as follows: $p^{(1)}$ is the first-order pressure; $\bar{p}^{(2)}$ is the mean second-order pressure (which is real and negative); $p_1^{(2)}$ is the second-order oscillatory pressure contributed by quadratic products of first-order quantities; and $p_2^{(2)}$ is the second-order oscillatory potential due to $\phi^{(2)}$. The cylinder is of radius a, in water of depth a, and $\nu a = 2$. In order to facilitate comparisons, the first- and second-order quantities are non-dimensionalized by $\rho g A$ and $\rho g A^2/a$ respectively (A being the wave amplitude). In figure 2 one observes a close agreement between the two sets of results for $p_2^{(2)}$. Moreover, it can be seen that $p_2^{(2)}$ is in fact larger than the other second-order quantities. Hence for estimation of second-order wave forces under these particular conditions, it would seem essential to include the contribution of $p_2^{(2)}$ in the analysis of the second-order oscillatory force.

Components of the wave elevation runup as a function of the azimuthal angle are plotted in figure 3, where the first-order and second-order quantities are non-dimensionalized by A and A^2/a respectively. The notation is similar to that used for the components of pressure. It is again observed that good agreement is obtained between the present results and those due to Kim & Yue. Moreover, the magnitude of the runup contributed by the second-order potential is also comparable with the other second-order quantities, and it shows larger variations along the waterline, especially between $\theta = 0°$ and $\theta = 90°$ (i.e. on the lee side).

In order to demonstrate the relative importance of the first- and second-order quantities, it is useful at this stage to provide a practical example demonstrating the significance of the second-order effects as the wave steepness ϵ increases. We consider a vertical cylinder of radius 12.5 m, which is typical of a column of a tension leg

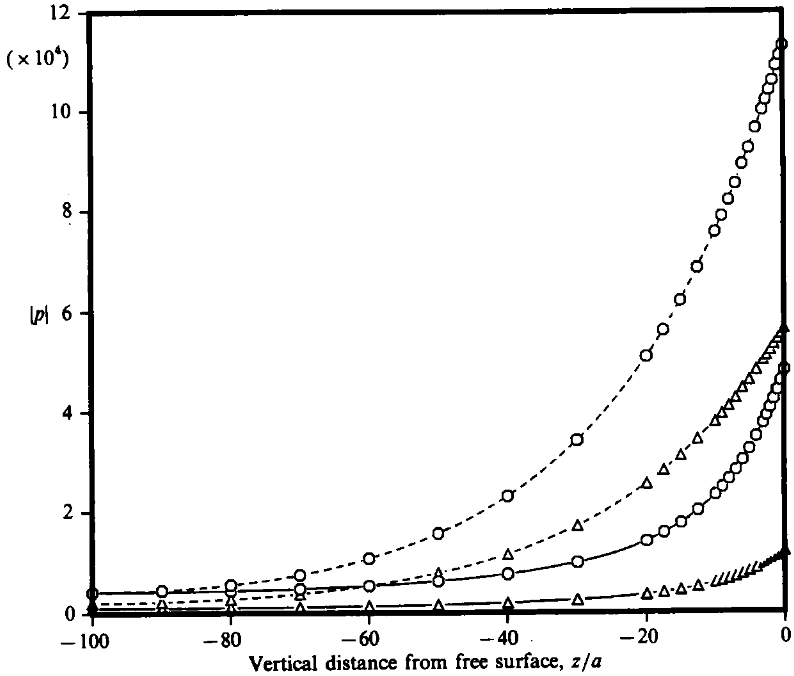


FIGURE 4. Components of pressure down the upwave generator ($\theta = 180°$) of a vertical cylinder of radius 12.5 m in a wave of 10 s in water of depth 100 m: --△--, $|p^{(1)}|$, $\epsilon = 0.05$; --○--, $|p^{(1)}|$, $\epsilon = 0.10$; —△—, $|p_2^{(2)}|$, $\epsilon = 0.05$; —○—, $|p_2^{(2)}|$, $\epsilon = 0.10$.

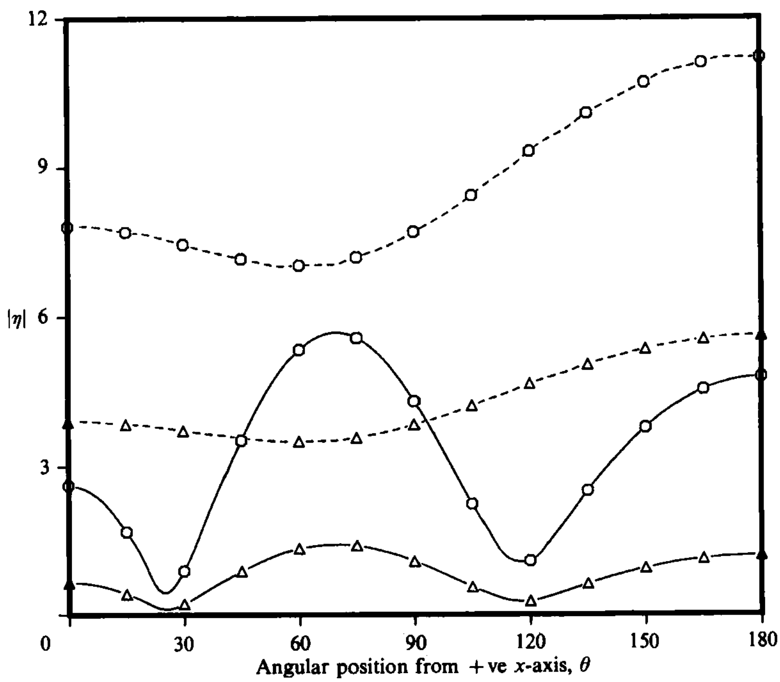


FIGURE 5. Components of wave elevation around the circumference of a vertical cylinder of radius 12.5 m in a wave a 10 s in water of depth 100 m: --△--, $|\eta^{(1)}|$, $\epsilon = 0.05$; --○--, $|\eta^{(1)}|$, $\epsilon = 0.10$; —△—, $|\eta_2^{(2)}|$, $\epsilon = 0.05$; —○—, $|\eta_2^{(2)}|$, $\epsilon = 0.10$.

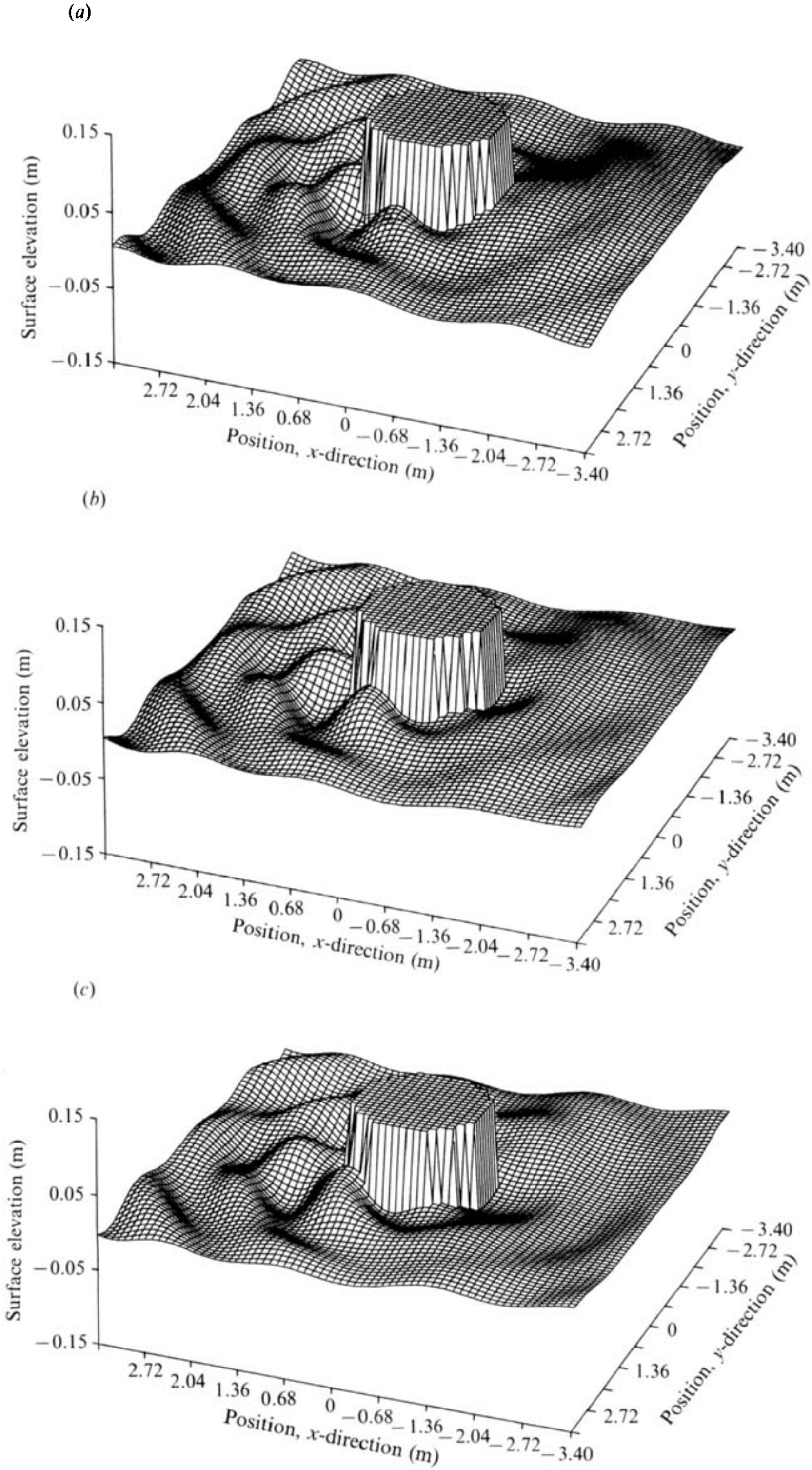
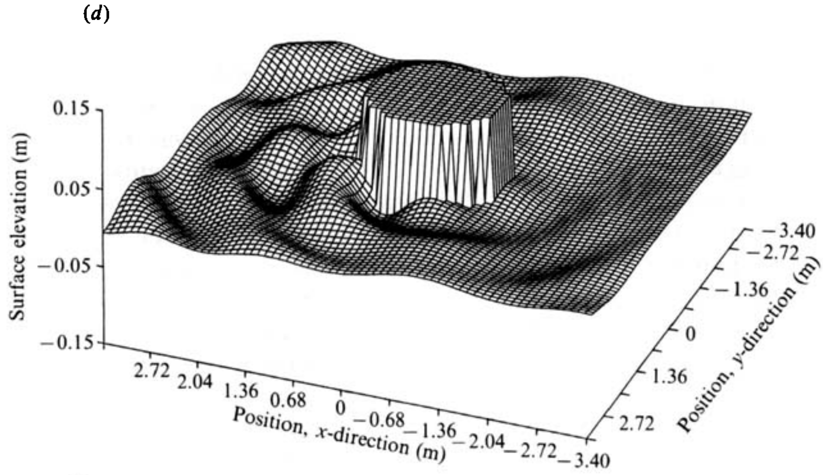


FIGURE 6(*a–c*). For caption facing page.



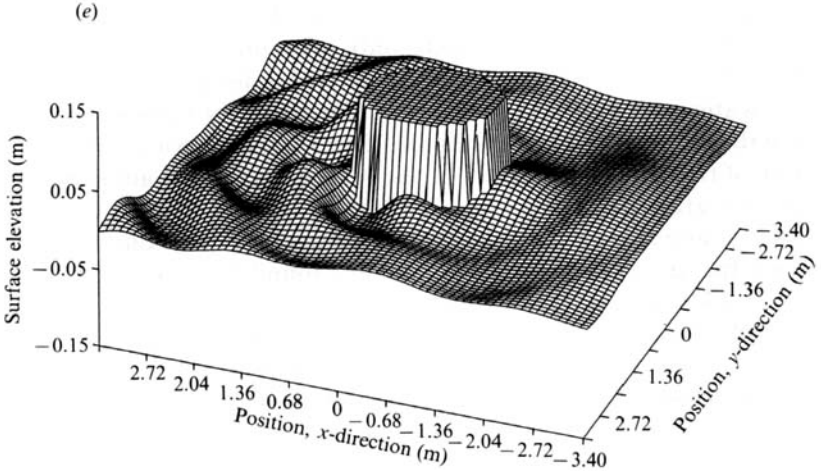


FIGURE 6. Isometrics of instantaneous dimensionless wave elevation $\mathrm{Re}[\eta_2^{(2)}\,\mathrm{e}^{-2\mathrm{i}\sigma t}]$ due to the second-order potential) near a vertical cylinder ($d/a = 1$, $ka = 1.4$), at intervals of one tenth of a wave period. (*a*) $t = 0$; (*b*) $t = 0.1T$; (*c*) $t = 0.2T$; (*d*) $t = 0.3T$; (*e*) $t = 0.4T$.

platform. For such a cylinder standing in water of depth 100 m and subjected to a 10 s wave, the vertical distributions of the magnitudes of the first-order pressure $|p^{(1)}|$ and the second-order pressure due to the second potential $|p_2^{(2)}|$, along the upwave side of the cylinder, are shown in figure 4. Results corresponding to two different wave steepnesses, $\epsilon = 0.05$ and a fairly steep wave $\epsilon = 0.1$, are superimposed on the same diagram. (Here ϵ is defined as the wave height divided by the wave length.) These results show the importance of the second-order effect in the steeper wave. Similar results for the runup are given in figure 5. They also make the obvious point that despite being only a correction term, the second-order effects which are proportional to the square of the wave elevation become progressively more important in higher sea states. In an irregular sea state, moreover, the incident wave system consists of a continuous spectrum of frequency components (e.g. described by the JONSWAP spectrum). In the frequency band well above the peak frequency of the spectrum, the contribution of the second-order component may be substantially higher than that due to the first-order component. This is because the pair of primary wave frequencies associated with the second-order quantities may lie in a frequency band near the peak frequency; whereas the first-order component being well above the peak frequency, the energy in the incident wave is hardly significant.

Useful qualitative information about the second-order behaviour can also be obtained through isometric plots of the instantaneous second-order free-surface profiles. Such plots are shown in figure 6 for the same cylinder ($d/a = 1$) used in the analysis leading to figures 2 and 3. The wave amplitude is one tenth of the radius and the axes of the isometrics are non-dimensionalized by the radius. The results are presented at intervals of one tenth of the wave period T for the case $ka = 1.4$. This interval corresponds to one fifth of the period of oscillation of the second-order quantities. The time is taken as zero when the crest of the incident wave has reached one quarter of a wavelength beyond the origin of spatial coordinates. The appearance of series of short curved waves at the downwave side of the cylinder is particularly noticeable.

Several interesting features are revealed by the second-order results, and among them the vertical variation of the potential on the body surface has particular physical importance. Because details concerning this behaviour have been discussed extensively in Eatock Taylor *et al.* (1989), only a summary is given here.

(*a*) The magnitude of the order diffracted potential decays slowly and may even increase with water depth. This implies that the resulting hydrodynamic pressure can penetrate deeper into the water column than the linear wave-induced pressure. It is indicative of the slow convergence of the corresponding second-order wave force with increase of water depth.

(*b*) The rate of decay with depth of the magnitude of the second-order diffracted potential down the upwave side of the cylinder is found to be less rapid than on the downwave side. This phenomenon seems to be consistent with the behaviour of the second-order diffracted potential in the far field. It is therefore reasonable to assume that a connection exists between the far-field behaviour of the free-surface boundary condition and the near-field solution (as has recently been demonstrated by Newman 1990).

(*c*) Owing to the significant influence of the second-order incident potential, the behaviour of the second-order diffracted potential in shallow water is substantially different from that in deep water. Moreover, the depth-dependence is much less pronounced at higher frequencies.

(*d*) Compared with the first-order potential, it has been found that the second-order potential varies rapidly in the azimuthal direction near the free surface. At greater depths, however, the azimuthal variation becomes more gradual, and similar to that of the first-order potential.

Finally we note that the results obtained by Kim & Yue (1989), using a numerical method developed for axisymmetric bodies, are in good agreement with the analytical solution presented here. This analytical solution has also been used to validate a numerical procedure for second-order diffraction by arbitrary three-dimensional bodies, as described by Chau (1989).

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Appendix A. Discussion of the free-surface boundary condition

On substituting the expression for $Z_n(\kappa_n z)$ and Z_{3n} from (4.9) and (5.6) respectively, and denoting

$$W_m(\kappa_n r) = I_m(\kappa_n r) - \frac{I'_m(\kappa_n a)}{K'_m(\kappa_n a)} K_m(\kappa_n r) \tag{A 1}$$

one can rewrite (5.2) as

$$\phi_s^{(2)}(\bar{r}_0) = \sum_{m=0}^{\infty} \phi_{s,m}^{(2)}(r_0, z_0) \cos(m\theta_0), \tag{A 2}$$

where

$$\begin{aligned}\phi_{s,m}^{(2)}(r_0, z_0) = \epsilon_m \sum_{n=0}^{\infty} \frac{4\kappa_n \cos(\kappa_n d)\cos(\kappa_n(z_0+d))}{\sin(2\kappa_n d)+2\kappa_n d} \int_a^{\infty} r\,\mathrm{d}r\, F_m(r) K_m(\kappa_n r_>)\, W_m(\kappa_n r_<) \\ -\epsilon_m D'_m\Big|_{r=a} \sum_{n=0}^{\infty} \frac{4\kappa_n \cos(\kappa_n d)\cos(\kappa_n(z_0+d))}{\sin(2\kappa_n d)+2\kappa_n d} \\ \times\left[\left(\frac{2k\sinh(2kd)-4\nu\cosh(2kd)}{4k^2+\kappa_n^2}\right)\frac{K_m(\kappa_n r_0)}{\kappa_n K'_m(\kappa_n a)}\right].\end{aligned} \tag{A 3}$$

Division of the above integral into two, i.e. from a to r_0 and from r_0 to infinity, gives

$$\begin{aligned}\int_a^{\infty} r\,\mathrm{d}r\, F_m(r) K_m(\kappa_n r_>)\, W_m(\kappa_n r_<) \\ = K_m(\kappa_n r_0)\int_a^{r_0} r\,\mathrm{d}r\, F_m(r)\, W_m(\kappa_n r) + W_m(\kappa_n r_0)\int_{r_0}^{\infty} r\,\mathrm{d}r\, F_m(r) K_m(\kappa_n r).\end{aligned} \tag{A 4}$$

By use of Bessel's equation, there follows

$$W_m(\kappa_n r) = \frac{1}{\kappa_n^2}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2}\right) W_m(\kappa_n r), \tag{A 5}$$

$$K_m(\kappa_n r) = \frac{1}{\kappa_n^2}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2}\right) K_m(\kappa_n r). \tag{A 6}$$

Substituting the above expressions into (A 4), and integrating by parts twice, one obtains

$$\begin{aligned}&\int_a^{\infty} r\,\mathrm{d}r\, F_m(r) K_m(\kappa_n r_>)\, W_m(\kappa_n r_<) \\ &= \frac{K_m(\kappa_n r_0)}{\kappa_n^2}[F_m(r)\, r\kappa_n W'_m(\kappa_n r) - F'_m(r)\, r W_m(\kappa_n r)]_{r=a}^{r=r_0} \\ &+ \frac{W_m(\kappa_n r_0)}{\kappa_n^2}[F_m(r)\, r\kappa_n K'_m(\kappa_n r) - F'_m(r)\, r K_m(\kappa_n r)]_{r=r_0}^{r\to\infty} \\ &+ \frac{K_m(\kappa_n r_0)}{\kappa_n^2}\int_a^{r_0} r\,\mathrm{d}r\, W_m(\kappa_n r)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2}\right) F_m(r) \\ &+ \frac{W_m(\kappa_n r_0)}{\kappa_n^2}\int_{r_0}^{\infty} r\,\mathrm{d}r\, K_m(\kappa_n r)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2}\right) F_m(r).\end{aligned} \tag{A 7}$$

Inserting the lower and upper limits into the terms in the square brackets, and making use of the relationships

$$W'_m(\kappa_n a) = 0, \tag{A 8}$$

$$W_m(\kappa_n a) = \frac{-1}{a\kappa_n K'_m(\kappa_n a)}, \tag{A 9}$$

$$K_m(\kappa_n r_0)\, W'_m(\kappa_n r_0) - W_m(\kappa_n r_0)\, K'_m(\kappa_n r_0) = \frac{1}{\kappa_n r_0}, \tag{A 10}$$

one can now write (A 7) in the form

$$\int_a^\infty r\,\mathrm{d}r\, F_m(r)\, K_m(\kappa_n r_>)\, W_m(\kappa_n r_<) = \frac{1}{\kappa_n^2}\left[F_m(r_0) - \frac{F'_m(a)}{\kappa_n}\frac{K_m(\kappa_n r_0)}{K'_m(\kappa_n a)}\right.$$
$$\left. + \int_a^\infty r\,\mathrm{d}r\, K_m(\kappa_n r_>)\, W_m(\kappa_n r_<)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2}\right) F_m(r)\right]. \tag{A 11}$$

Consequently, if similar procedures are applied repeatedly to the integral on the right of (A 11), one obtains

$$\int_a^\infty r\,\mathrm{d}r\, F_m(r)\, K_m(\kappa_n r_>)\, W_m(\kappa_n r_<)$$
$$= \frac{1}{\kappa_n^2}\left[F_m(r_0) - \frac{F'_m(a)}{\kappa_n}\frac{K_m(\kappa_n r_0)}{K'_m(\kappa_n a)} + \text{higher order terms in } \frac{1}{\kappa_n}\right]. \tag{A 12}$$

Substituting (A 12) into (A 3) yields

$$\phi^{(2)}_{\mathrm{s},m}(r_0, z_0) = \epsilon_m \sum_{n=0}^{\infty} \frac{4\cos(\kappa_n d)\cos(\kappa_n(z_0+d))}{\kappa_n(\sin(2\kappa_n d) + 2\kappa_n d)}\left[F_m(r_0) - \frac{F'_m(a)}{\kappa_n}\frac{K_m(\kappa_n r_0)}{K'_m(\kappa_n a)}\right.$$
$$\left. - \kappa_n^2 D'_m\Big|_{r=a}\left(\frac{2k\sinh(2kd) - 4\nu\cosh(2kd)}{4k^2+\kappa_n^2}\right)\frac{K_m(\kappa_n r_0)}{\kappa_n K'_m(\kappa_n a)} + \text{higher order terms in } \frac{1}{\kappa_n}\right]. \tag{A 13}$$

Now

$$\frac{\partial \phi^{(2)}_{\mathrm{s},m}}{\partial z_0} - 4\nu\,\phi^{(2)}_{\mathrm{s},m}$$
$$= \epsilon_m \sum_{n=0}^{\infty}\left(\frac{-4\cos(\kappa_n d)\sin(\kappa_n(z_0+d))}{\sin(2\kappa_n d) + 2\kappa_n d} - \frac{16\nu\cos(\kappa_n d)\cos(\kappa_n(z_0+d))}{\kappa_n(\sin(2\kappa_n d)+2\kappa_n d)}\right)\times[\ldots], \tag{A 14}$$

where [...] denotes the terms inside the square bracket of (A 13). By use of

$$4\nu\cos(\kappa_n d) = -\kappa_n \sin(\kappa_n d) \tag{A 15}$$

(A 14) may then be reduced to

$$\frac{\partial \phi^{(2)}_{\mathrm{s},m}}{\partial z_0} - 4\nu\,\phi^{(2)}_{\mathrm{s},m} = \epsilon_m \sum_{n=0}^{\infty} \frac{4\sin(-\kappa_n z_0)}{\sin(2\kappa_n d) + 2\kappa_n d}[\ldots]. \tag{A 16}$$

For large values of n, $\kappa_n d \to n\pi$; from this and the asymptotic expression for the Bessel function K_m, the terms inside the square bracket are found to have the form

$$\left[F_m(r_0) - \frac{d}{n\pi}\left(\frac{a}{r_0}\right)^{\frac{1}{2}} \exp\left(\frac{-n\pi}{d}(r_0 - a)\right)\right.$$
$$\times \{F'_m(a) + D'_m\Big|_{r=a} (2k \sinh(2kd) - 4\nu \cosh(2kd))\}$$
$$\left. + \text{higher order terms in } \left(\frac{d}{n\pi}\right)\right] \sim F_m(r_0) \quad \text{for large } n. \tag{A 17}$$

Substitution into (A 16) yields

$$\frac{\partial \phi^{(2)}_{s,m}}{\partial z_0} - 4\nu\, \phi^{(2)}_{s,m} = \epsilon_m \sum_{n=0}^{N-1} \frac{4 \sin(-\kappa_n z_0)}{\sin(2\kappa_n d) + 2\kappa_n d}[\ldots] + \epsilon_m F_m(r_0) \sum_{n=N}^{\infty} \frac{4 \sin(-n\pi z_0/d)}{2n\pi}, \tag{A 18}$$

where N is sufficiently large. Hence

$$\lim_{z_0 \to 0}\left\{\frac{\partial \phi^{(2)}_{s,m}}{\partial z_0} - 4\nu\, \phi^{(2)}_{s,m}\right\} = \frac{2}{\pi}\epsilon_m F_m(r_0) \sum_{n=N}^{\infty} \frac{\sin(-n\pi z_0/d)}{n}. \tag{A 19}$$

Since $z_0 < 0$, it is clear that $0 < (-\pi z_0/d) < 2\pi$. Hence from Jolley (1961)

$$\lim_{z_0 \to 0} \sum_{n=N}^{\infty} \frac{\sin[n(-\pi z_0/d)]}{n} = \lim_{z_0 \to 0} \tfrac{1}{2}(\pi + \pi z_0/d) = \tfrac{1}{2}\pi. \tag{A 20}$$

Substituting (A 20) into (A 19), one obtains

$$\lim_{z_0 \to 0}\left\{\frac{\partial \phi^{(2)}_{s,m}}{\partial z_0} - 4\nu\, \phi^{(2)}_{s,m}\right\} = \frac{2}{\pi}\epsilon_m F_m(r_0)\, \tfrac{1}{2}\pi$$
$$= \epsilon_m F_m(r_0), \tag{A 21}$$

or

$$\lim_{z_0 \to 0}\left\{\frac{\partial \phi^{(2)}_{s}}{\partial z_0} - 4\nu\, \phi^{(2)}_{s}\right\} = \sum_{m=0}^{\infty} \epsilon_m F_m(r_0) \cos(m\theta_0)$$
$$= F(r_0, \theta_0). \tag{A 22}$$

This demonstrates that the inhomogeneous free-surface boundary condition is in fact satisfied by the solution of $\phi^{(2)}_s$ given in (5.2).

Appendix B. Evaluation of the infinite integral

The infinite integral on the right-hand side of (6.14) can be written as

$$I_{F^2} = \tfrac{1}{2}\pi \mathrm{i}\, Z_0(\kappa z_0) Z_0(0) \sum_{m=0}^{\infty} \epsilon_m \cos(m\theta_0)\left[J_m(\kappa r_0) - \frac{J'_m(\kappa a)}{H'_m(\kappa a)} H_m(\kappa r_0)\right]$$
$$\times \int_{r_s}^{\infty} r\,\mathrm{d}r \sum_{q=-\infty}^{\infty}\left\{\frac{\mathrm{i}\sigma}{2g} k^2 (3\tanh^2(kd) - 1)[S_{m-q} S_q + 2T_{m-q} S_q]\right.$$
$$\left. + \frac{\mathrm{i}\sigma}{g}\left[\frac{\partial}{\partial r} S_{m-q} \frac{\partial}{\partial r} S_q + 2\frac{\partial}{\partial r} T_{m-q} \frac{\partial}{\partial r} T_q\right]\right\} H_m(\kappa r). \tag{B 1}$$

In this form, contributions from the evanescent modes and the circumferential derivatives at large values of r have been neglected. Integrating by parts three times and using the Bessel differential equation, one can show that the integral terms containing the r-derivatives may be replaced by new integrals which are free from derivatives, plus some residual terms evaluated at r_s. This has been found advantageous for the purpose of numerical implementation. The resulting expression for I_{F^2} can then be written as

$$I_{F^2} = \frac{-\pi\sigma}{4g} Z_0(\kappa z_0) Z_0(0) \sum_{m=0}^{\infty} \epsilon_m \cos(m\theta_0) \left[J_m(\kappa r_0) - \frac{J'_m(\kappa a)}{H'_m(\kappa a)} H_m(\kappa r_0)\right] I_{F_\infty}$$
$$+ \frac{\pi\sigma}{4g} Z_0(\kappa z_0) Z_0(0) \sum_{m=0}^{\infty} \epsilon_m \cos(m\theta_0) \left[J_m(\kappa r_0) - \frac{J'_m(\kappa a)}{H'_m(\kappa a)} H_m(\kappa r_0)\right]$$
$$\times \sum_{q=-\infty}^{\infty} r_s \left\{\left[S_{m-q} H_m \frac{\partial}{\partial r} S_q + S_q H_m \frac{\partial}{\partial r} S_{m-q} - S_{m-q} S_q \frac{\partial}{\partial r} H_m\right]\right.$$
$$\left.+ 2\left[T_{m-q} H_m \frac{\partial}{\partial r} S_q + S_q H_m \frac{\partial}{\partial r} T_{m-q} - T_{m-q} S_q \frac{\partial}{\partial r} H_m\right]\right\}_{r=r_s} \qquad \text{(B 2)}$$

where

$$I_{F_\infty} = (3 \tanh^2(kd) + k^2 - \kappa^2) \int_{r_s}^{\infty} r \, \mathrm{d}r \sum_{q=-\infty}^{\infty} [S_{m-q} S_q + 2T_{m-q} S_q] H_m(\kappa r). \qquad \text{(B 3)}$$

To evaluate the resulting integrals which involve triple products of Hankel functions, Hankel's asymptotic expansions seem to be the most convenient starting point. As defined by Abramowitz & Stegun (1972)

$$H_m(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} [P_m(x) + \mathrm{i}Q_m(x)] \, \mathrm{e}^{\mathrm{i}(x-\gamma_m)}, \qquad \text{(B 4)}$$

where

$$\gamma_m = \tfrac{1}{2}m\pi + \tfrac{1}{4}\pi, \qquad \text{(B 5)}$$

$$P_m(x) = \sum_{\iota=0}^{\infty} \frac{(-1)^\iota \Gamma(\frac{1}{2} + m + 2\iota)}{(2\iota)! \, \Gamma(\frac{1}{2} + m - 2\iota) (2x)^{2\iota}}, \qquad \text{(B 6)}$$

$$Q_m(x) = \sum_{\iota=0}^{\infty} \frac{(-1)^\iota \Gamma(\frac{3}{2} + m + 2\iota)}{(2\iota+1)! \, \Gamma(-\frac{1}{2} + m - 2\iota) (2x)^{2\iota+1}}. \qquad \text{(B 7)}$$

For the purpose of calculating the integral from r_s to infinity, the asymptotic series $P_m(x)$ and $Q_m(x)$ must first be truncated at some finite number of terms. It is known that whenever $\iota > \frac{1}{2}m$, the remainder after ι terms in the expansion of $P_m(x)$ will not exceed the $(\iota+1)$th term in absolute value. Similar considerations hold for $Q_m(x)$. A simple rule for obtaining a good estimate can therefore be achieved by truncating $P_m(x)$ and $Q_m(x)$ at

$$\iota = M_m = \tfrac{1}{2}m + 2. \qquad \text{(B 8)}$$

The various terms in the asymptotic series may now be obtained as follows. We use (B 2) in the form

$$H_m(kr) = \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(kr-\gamma_m)} \sum_{\iota=0}^{M_m} C_{m\iota} (\kappa\rho)^{-\iota} \qquad \text{(B 9)}$$

where

$$C_{m0} = 1 \qquad \text{(B 10)}$$

$$C_{m\iota} = \frac{\mathrm{i}^\iota (4m^2-1)(4m^2-3)\ldots(4m^2-(2\iota-1)^2)}{\iota! \, 8^\iota}. \qquad \text{(B 11)}$$

A recursive relationship for $C_{m\iota}$ can readily be derived and given as

$$C_{m,\iota+1} = \frac{\mathrm{i}(4m^2-(2\iota+1)^2)}{8(\iota+1)}C_{m\iota}. \tag{B 12}$$

Integral (6.16), which is based on the first part of the integrand in (B 3), may now be written as a triple sum

$$I^{(1)}_{\iota mn} = D^{(1)}_{\iota mn} \sum_{p,q,s}^{\infty} C_{\iota p} C_{mq} C_{ns} k^{-(p+q)} \kappa^{-s} \int_{r_s}^{\infty} \mathrm{d}r \frac{\mathrm{e}^{\mathrm{i}(2k+\kappa)r}}{r^{p+q+s+1/2}}, \tag{B 13}$$

where
$$D^{(1)}_{\iota mn} = \frac{2}{\pi k}\left(\frac{2}{\pi\kappa}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}(\gamma_\iota+\gamma_m+\gamma_n)}. \tag{B 14}$$

Similarly (6.17) becomes

$$I^{(2)}_{\iota mn} = D^{(2)}_{\iota mn} \sum_{p,q,s}^{\infty} (-1)^p C_{\iota p} C_{mq} C_{ns} k^{-(p+q)} \kappa^{-s} \int_{r_s}^{\infty} \mathrm{d}r \frac{\mathrm{e}^{\mathrm{i}\kappa r}}{r^{p+q+s+1/2}}, \tag{B 15}$$

where
$$D^{(2)}_{\iota mn} = \frac{2}{\pi k}\left(\frac{2}{\pi\kappa}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}(-\gamma_\iota+\gamma_m+\gamma_n)}. \tag{B 16}$$

The integrals in (B 13) and (B 15) are of the form

$$A_n = \int_{r_s}^{\infty} \mathrm{d}r \frac{\mathrm{e}^{\mathrm{i}\alpha r}}{r^{n+\frac{1}{2}}}, \tag{B 17}$$

with $\alpha = 2k+\kappa$ or $\alpha = \kappa$. They can now be evaluated by the method of asymptotic expansion of integrals. Integration by parts is employed to derive the recurrence formula

$$A_n = \frac{\mathrm{i}}{\alpha}\frac{\mathrm{e}^{\mathrm{i}\alpha r_s}}{r_s^{n+\frac{1}{2}}} + \frac{n+\frac{1}{2}}{\mathrm{i}\alpha}A_{n+1}, \tag{B 18}$$

which readily leads to the power series

$$A_n = -\sum_{j=1}^{N} \frac{(n+j-\frac{3}{2})!}{(n-\frac{1}{2})!}\frac{1}{(\mathrm{i}\alpha)^j}\frac{\mathrm{e}^{\mathrm{i}\alpha r_s}}{r^{n+j-\frac{1}{2}}} + E_N. \tag{B 19}$$

The truncation error E_N is given as

$$E_N = \frac{(n+N-\frac{1}{2})!}{(n-\frac{1}{2})!}\frac{1}{(\mathrm{i}\alpha)^N}\int_{r_s}^{\infty} \mathrm{d}r \frac{\mathrm{e}^{\mathrm{i}\alpha r}}{r^{n+N+\frac{1}{2}}}. \tag{B 20}$$

One therefore has

$$|E_N| \leqslant \left| \frac{(n+N-\frac{1}{2})!}{(n-\frac{1}{2})!}\frac{1}{(\mathrm{i}\alpha)^N}\int_{r_s}^{\infty} \mathrm{d}r \frac{1}{r^{n+N+\frac{1}{2}}} \right|$$

$$= \frac{(n+N-\frac{3}{2})!}{(n-\frac{1}{2})!}\frac{1}{\alpha^N}\frac{1}{r_s^{n+N-\frac{1}{2}}}. \tag{B 21}$$

From this it may be concluded that the truncation error always has the same order of magnitude as the last remaining term in the series (B 19). With this in mind, one can formulate a simple rule for obtaining good numerical results by considering the ratio of the $(N+1)$th term in (B 19) to the Nth term, i.e.

$$\left| \frac{(N+1)\text{th term}}{N\text{th term}} \right| = \frac{n+N-\frac{1}{2}}{\alpha r_s}. \tag{B 22}$$

The successive terms decrease steadily in magnitude as long as $n+N-\frac{1}{2} < \alpha r_s$, but increase unboundedly in magnitude with increasing N. The best estimate would then be obtained by choosing N as

$$N = \text{greatest integer less than } [\alpha r_s - n]. \tag{B 23}$$

Appendix C. The nature of the singularity at the free surface

The following discussion is based on some ideas presented by Fenton (1978). We note first, by examination of the integrand in the expression (5.2) for $\phi_s^{(2)}$, that all terms associated with $n = 0$ are finite. This is not the case however for the corresponding terms with $n > 0$. To see this, let

$$L_{mn_{n>0}} = Z_n(\kappa_n z_0)\, Z_n(0)\, K_m(\kappa_n r_>)\left[I_m(\kappa_n r_<) - \frac{I'_m(\kappa_n a)}{K'_m(\kappa_n a)} K_m(\kappa_n r_<)\right]. \tag{C 1}$$

Substitution of the expression for $Z_n(\kappa_n z)$ given in (4.9), gives

$$L_{mn_{n>0}} = \frac{-2(\kappa_n^2+16\nu^2)}{-d(\kappa_n^2+16\nu^2)+4\nu} \cos(\kappa_n(z_0+d)) \cos(\kappa_n d)$$
$$\times K_m(\kappa_n r_>)\left[I_m(\kappa_n r_<) - \frac{I'_m(\kappa_n a)}{K'_m(\kappa_n a)} K_m(\kappa_n r_<)\right]. \tag{C 2}$$

For large n, one can show that

$$\kappa_n = \frac{n\pi}{d} + O\left(\frac{1}{n}\right). \tag{C 3}$$

Hence it can be deduced that

$$\frac{-2(\kappa_n^2+16\nu^2)}{-d(\kappa_n^2+16\nu^2)+4\nu} \approx \frac{2}{d} + O\left(\frac{1}{n^2}\right), \tag{C 4}$$

and

$$\cos(\kappa_n(z_0+d)) \cos(\kappa_n d) \approx \cos n(\pi z_0/d). \tag{C 5}$$

Recalling the asymptotic expansions for I_m and K_m given in Abramowitz & Stegun (1972), i.e.

$$I_m(x) \approx \frac{e^x}{(2\pi x)^{\frac{1}{2}}} \quad \text{for large } x, \tag{C 6}$$

$$K_m(x) \approx (\pi/2x)^{\frac{1}{2}} e^{-x} \quad \text{for large } x, \tag{C 7}$$

one sees that the products of the Bessel functions behave as

$$K_m(\kappa_n r_>)\left[I_m(\kappa_n r_<) - \frac{I'_m(\kappa_n a)}{K'_m(\kappa_n a)} K_m(\kappa_n r_<)\right]$$
$$\approx \frac{d}{2n\pi} \frac{1}{(r_> r_<)^{\frac{1}{2}}} \left[\exp\left[\frac{-n\pi}{d}(r_> - r_<)\right] + \exp\left[\frac{-n\pi}{d}(r_> + r_< - 2a)\right]\right]. \tag{C 8}$$

Defining the asymptotic form of L_{mn} as $n \to \infty$ by L^*_{mn}, one has therefore

$$L^*_{mn} \approx \frac{1}{\pi} \frac{\cos[n(\pi z_0/d)]}{n} \frac{1}{(r_> r_<)^{\frac{1}{2}}} \left[\exp\left[\frac{-n\pi}{d}(r_> - r_<)\right] + \exp\left[\frac{-n\pi}{d}(r_> + r_< - 2a)\right]\right]. \tag{C 9}$$

Hence

$$\sum_{n=1}^{\infty} L^*_{mn} \approx \frac{1}{\pi}\frac{1}{(r_> r_<)^{\frac{1}{2}}}\sum_{n=1}^{\infty} \exp\left[\left(\frac{-n\pi}{d}\right)(r_> - r_<)\right]\frac{\cos\,[n(\pi z_0/d)]}{n}$$

$$+\frac{1}{\pi}\frac{1}{(r_> r_<)^{\frac{1}{2}}}\sum_{n=1}^{\infty} \exp\left[\left(\frac{-n\pi}{d}\right)(r_> + r_< - 2a)\right]\frac{\cos\,[n(\pi z_0/d)]}{n}. \quad \text{(C 10)}$$

The infinite sums can be obtained from Jolley (1961), leading to

$$\sum_{n=1}^{\infty} L^*_{mn} \approx \frac{-1}{2\pi}\frac{1}{(r_> r_<)^{\frac{1}{2}}}\ln\left[1-2\exp\left[\frac{-\pi}{d}(r_> - r_<)\right]\cos\,(\pi z_0/d)+\exp\left[\frac{-2\pi}{d}(r_> - r_<)\right]\right]$$

$$\frac{-1}{2\pi}\frac{1}{(r_> r_<)^{\frac{1}{2}}}\ln\left[1-2\exp\left[\frac{-\pi}{d}(r_> + r_< - 2a)\right]\cos\,(\pi z_0/d)+\exp\left[\frac{-2\pi}{d}(r_> + r_< - 2a)\right]\right]. \quad \text{(C 11)}$$

Except for the case $r_0 = a$, it is the first term which contributes as $z_0 \to 0$, $r \to r_0$. For $r_0 = a$

$$\sum_{n=1}^{\infty} L^*_{mn} \approx \frac{1}{\pi}\frac{1}{(ra)^{\frac{1}{2}}}\ln\left[1-2\exp\left[\frac{-\pi}{d}(r-a)\right]\cos\,(\pi z_0/d)+\exp\left[\frac{-2\pi}{d}(r-a)\right]\right]. \quad \text{(C 12)}$$

One now expands the exponential function and cosine function in powers of their arguments in (C 11) and (C 12), which as $r_> \to r_<$, $z_0 \to 0$, gives for $r_0 \neq a$

$$\sum_{n=1}^{\infty} L^*_{mn} \approx \frac{-1}{2\pi}\frac{1}{(r_> r_<)^{\frac{1}{2}}}\ln\left[\frac{\pi^2}{d^2}(z^2+(r_> - r_<)^2)\right]; \quad \text{(C 13)}$$

for $r_0 = a$

$$\sum_{n=1}^{\infty} L^*_{mn} \approx \frac{-1}{\pi}\frac{1}{(ra)^{\frac{1}{2}}}\ln\left[\frac{\pi^2}{d^2}(z^2+(r-a)^2)\right]. \quad \text{(C 14)}$$

The resulting expressions show the logarithmic nature of the singularity, and hence the infinite series (for n) in the integrand of (5.2) does not converge at some points.

The efficiency of the numerical integration can, however, be improved by subtracting out the singularities. To illustrate this, let us consider the simplest case with $r_0 = a$ and $z_0 = 0$, and write

$$\int_a^{r_s} r\,\mathrm{d}r F_m(r)\sum_{n=1}^{\infty} L_{mn} = \int_a^{r_s} r\,\mathrm{d}r F_m(r)\sum_{n=1}^{\infty}\{L_{mn} - L^*_{mn}\} + \int_a^{r_s} r\,\mathrm{d}r F_m(r)\sum_{n=1}^{\infty} L^*_{mn}. \quad \text{(C 15)}$$

The contribution for $r > r_s$ is evaluated using (B 2). Substituting (C 14) into the last integral, one obtains

$$\int_a^{r_s} r\,\mathrm{d}r F_m(r)\sum_{n=1}^{\infty} L_{mn}$$

$$= \int_a^{r_s} r\,\mathrm{d}r F_m(r)\sum_{n=1}^{\infty}\{L_{mn} - L^*_{mn}\} - \frac{1}{\pi}\int_a^{r_s} r\,\mathrm{d}r F_m(r)\frac{1}{(ra)^{\frac{1}{2}}}\ln\left[\frac{\pi^2}{d^2}(r-a)^2\right]. \quad \text{(C 16)}$$

The first integral in the above expression is non-singular and can therefore be evaluated using Gaussian quadrature as before. The second integral, however, has a logarithmic singularity and may be evaluated by a special numerical scheme with $\ln(\xi)$ weighting coefficients.

An alternative method of performing the second integral in (C 16) is also possible, which relies on a nonlinear transformation whose Jacobian smoothes out the singularity (Telles 1987). One rewrites the second integral in (C 16) as

$$I = \int_a^{r_s} \mathrm{d}r\, P_m(r) \ln\left[\frac{\pi}{d}(r-a)\right], \tag{C 17}$$

where

$$P_m(r) = \frac{-2}{\pi}(r/a)^{\frac{1}{2}} F_m(r). \tag{C 18}$$

By means of the linear transformation

$$r = \tfrac{1}{2}(r_s - a)\,\xi + \tfrac{1}{2}(r_s + a), \tag{C 19}$$

the integral can be written as

$$I = \tfrac{1}{2}(r_s - a)\int_{-1}^{1} \mathrm{d}\xi\, P_m(\xi) \ln\left[\tfrac{1}{2}(r_s - a)(\pi/d)(\xi+1)\right]. \tag{C 20}$$

If a second-order transformation is then chosen such that

$$\xi = \tfrac{1}{2}\eta^2 + \eta - \tfrac{1}{2}, \tag{C 21}$$

expression (C 20) can be written as

$$I = \tfrac{1}{2}(r_s - a)\int_{-1}^{1} \mathrm{d}\eta\,(\eta+1)\, P_m(\eta) \ln\left[\tfrac{1}{2}(r_s - a)(\pi/d)(\tfrac{1}{2}\eta^2 + \eta + \tfrac{1}{2})\right]. \tag{C 22}$$

Standard Gaussian integration can now be employed for the evaluation of (C 22) because the Jacobian cancels the logarithmic singularity at $\eta = -1$ (i.e. at $r = a$ in the original integral). This alterative procedure has not, however, been implemented in the computer program used to generate the results presented in the paper.

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